JOURNAL OF ALGEBRA **200,** 118–133 (1998)
ARTICLE NO. JA977211

Butler Theory over Murley Groups

Ulrich F. Albrecht and H. Pat Goeters

Department of Mathematics, Auburn University, 218 Parker Hall, Auburn, Alabama 36849

Received December 13, 1996

IN LOVING MEMORY OF IRMGARD ALBRECHT

One of the most widely studied classes of torsion-free abelian groups of finite rank is the class of classical Butler groups. A Butler group can be viewed either as a pure subgroup of a completely decomposable group of finite rank or an epimorphic image of such a group. Excellent sources for the basic properties of Butler groups are [9, 11, 12] of Arnold and Vinsonhaler. Vinsonhaler has also suggested an investigation of possible extensions to the theory of Butler groups to pure subgroups of $\mathscr{A}$-decomposable groups where $\mathscr{A}$ is a suitably chosen family of torsion-free groups of finite rank [12]. It is the goal of this paper to follow Vinsonhaler's suggestion.

Given any family $\mathscr{A}$ of torsion-free abelian groups of finite rank, a sequence $0 \to L \to M \to N \to 0$ is called *$\mathscr{A}$-balanced* if each member of $\mathscr{A}$ is projective with respect to it. A group M is called (*finitely*) *$\mathscr{A}$-decomposable* if $M \cong \bigoplus_{I \in I} A_i$ for some (*finite*) collection $\{A_i\}_{I \in I}$ from $\mathscr{A}$. Our definition of an *$\mathscr{A}$-Butler group* is a group G that is an $\mathscr{A}$-balanced epimorphic image of a finitely $\mathscr{A}$-decomposable group. When investigating $\mathscr{A}$-Butler groups, some limitations regarding the generality of $\mathscr{A}$ must be in place in order to avoid inordinate restrictions on our class of $\mathscr{A}$-Butler groups. In view of the discussion of $\mathscr{A}$-decomposable groups in [10, 18, 5] we expect such limitations on $\mathscr{A}$ to be in the form of some pseudo-rigidity conditions together with some conditions imposed on the endomorphism rings of the elements of $\mathscr{A}$.

Recall that the p-rank of a group G, $r_p(G)$, is $\dim_{\mathbb{Z}(p)} G/pG$, and a torsion-free group A is called a *Murley group* when $r_p(A) \leq 1$ for all

0021-8693/98 $25.00

primes p. The group A is called *irreducible* if A has no proper pure fully invariant subgroups. One of the more desirable features of a rank one group A is that any epimorphism $A^n \to B$, with B torsion-free and reduced, splits. Section 1 characterizes the torsion-free groups of finite rank having this property. To summarize Theorem 1.1, an indecomposable group A has the splitting property above precisely when A is an irreducible Murley group.

Several other important properties of the class of irreducible indecomposable Murley groups are explored. Since an irreducible indecomposable Murley group A is of the form $X \otimes E(A)$ for some subgroup X of $\mathbb{Q}$ and $E(A)^+$ is also an irreducible indecomposable Murley group, we restrict our discussion to the case that $\mathscr{A}$ is the class $\mathscr{A}_R$ of rank-1 modules over a ring R whose additive group is an irreducible indecomposable Murley group. However, this restriction is not dictated by technical reasons since Proposition 2.9 shows that this actually is the most general choice for $\mathscr{A}$ if we want to preserve the basic properties of Butler groups provided $\mathscr{A}$ contains a minimal element.

In Section 2 we describe the $\mathscr{A}_R$-Butler groups. Proposition 2.2 shows that the elements of $\mathscr{A}_R$ are of the form $X \otimes R$ where X is a rank-1 group, allowing us to establish various characterizations of Butler groups for $\mathscr{A}_R$-Butler groups (Theorem 2.7). Most surprisingly, the $\mathscr{A}_R$-Butler groups carry a natural R-module structure making them Butler R-modules. Our final result uses Theorem 2.7 to relate the structure of $\mathscr{A}_R$-Butler groups to that of classical Butler groups by showing that the former are precisely the groups of the form $R \otimes B$ for some classical Butler group B (Theorem 2.10).

1. ON THE EXISTENCE OF A-SOLVABLE GROUPS

Given an abelian group A, we define an adjoint pair (H_A, T_A) of functors between the categories of abelian groups and right $E(A)$-modules by $H_A(G) = \operatorname{Hom}(A, G)$ for an abelian group G and $T_A(M) = M \otimes_{E(A)} A$ for a right $E(A)$-module M. The natural maps associated with these functors are $\theta_G: T_A H_A(G) \to G$ and $\phi_M: M \to H_A T_A(M)$, respectively, which are defined by $\theta_G(\alpha \otimes a) = \alpha(a)$ and $[\phi_M(m)](a) = m \otimes a$ for all $\alpha \in H_A(G)$, $m \in M$, and $a \in A$. As always, the image of θ_G in G is denoted by $S_A(G)$.

The class of all *A-solvable groups* consists of those groups G for which θ_G is an isomorphism, while the (*finitely*) *A-generated groups* are those which are epimorphic images of a (finite) direct sum of copies of A. If A is torsion-free, we have seen in [2] that every torsion-free A-generated group is A-solvable if and only if A is homogeneous completely decomposable. In this section we address the question: Which torsion-free groups A have the property that the reduced (cotorsion-free) A-generated groups are A-solvable? Our discussion will focus on the class of strongly homogeneous groups, where a torsion-free group A is *strongly homogeneous* if, for any two rank-1 pure subgroups X and Y of A, there is an automorphism of A sending X onto Y.

Recall, the p-rank A will be denoted by $r_p(A)$. The endomorphism ring of an indecomposable Murley group is a principal ideal domain [17] and are therefore examples of *generalized rank* 1 *groups*—ie., abelian groups whose endomorphism rings are semi-prime, right and left Noetherian, and hereditary [1]. At this point we want to remind the reader that an indecomposable Murley group A is strongly homogeneous if and only if it is irreducible [9]. Moreover, if A is an irreducible indecomposable Murley group, then $r_p(A) = r_p(U)$ for all primes p and all pure rank-1 subgroups U of A, and so A/V is divisible for all non-zero pure subgroups V of A. Finally, recalling the terminology from [9], a group A is called *p-simple* for a given prime p, if $E(A)/pE(A)$ is a simple algebra. Below, A-balanced and A-generated mean $\{A\}$-balanced and $\{A\}$-generated.

THEOREM 1.1. *The following conditions are equivalent for a torsion-free abelian group A of finite rank*:

(a) *A is p-simple for all primes p for which $pA \neq A$, and $A \cong B_1^{n_1} \oplus \cdots \oplus B_m^{n_m}$ where each B_i is an irreducible indecomposable Murley group.*

(b) *Every pure-exact sequence $A^n \to G \to 0$ in which G is reduced, splits.*

(c) (i) *A is a generalized rank* 1 *group.*

(ii) *Every torsion-free reduced A-generated group is A-solvable.*

Proof. (a) $\Rightarrow$ (b). Let G be a torsion-free reduced group fitting into an exact sequence $A^n \xrightarrow{\beta} G \to 0$ for some $n < \omega$. Set $A_i = B_i^{n_i}$ and define G_i by the sequence $A_i^n \xrightarrow{\beta|_{A_i^n}} G_i \to 0$. Then $G = \sum_{i=1}^m G_i$. To show this sum is direct, suppose $x \in G_i \cap [\sum_{j\neq i} G_j]$. Because A is p-simple, given a prime p with $pA \neq A$, $pB_j \neq B_j$ for exactly one index j. So x has infinite p-height for all the primes p for which $B_i \neq pB_i$ as an element of $\sum_{j\neq i} G_j$. On the other hand, x also has infinite p-height for all the primes p which divide B_i since G_i is B_i-generated. Therefore, x belongs to the divisible subgroup of G and must be zero. Therefore it suffices to show that $(B_i^{n_i})^n \to G_i$ splits. In consequence, assume that A is an irreducible indecomposable Murley group. By the remarks preceding the theorem, A is a generalized rank 1 group. We first show that G is A-solvable.

Since G is A-generated, it remains to show that $\ker \theta_G = 0$. Choose $x \in \ker \theta_G$, and write $x = \sum_{i=1}^m \alpha_i \otimes a_i$ with $\alpha_1, \ldots, \alpha_m \in H_A(G)$ and

$a_1, \ldots, a_m \in A$. As mentioned prior to the theorem, A is strongly homogeneous. From the definition of strongly homogeneous, it is easy to check that $A = R \otimes X$ for some $X \subseteq \mathbb{Q}$ and $R = E(A)$. Consequently, there is a non-zero $a \in A$ and a non-zero integer k such that $kx = \phi \otimes a$ where $\phi = \sum_{i=1}^{m} \alpha_i r_i \in H_A(G)$. Since A is an indecomposable Murley group and G is torsion-free, $A/\ker\alpha$ is divisible or α is one-to-one for every $0 \neq \alpha \in \operatorname{Hom}(A, G)$. The former case cannot occur since G is reduced; hence all non-zero maps from A to G are one-to-one. In particular, $\phi(a) = 0$ yields $kx = 0$. We showed in [1] that every generalized rank 1 group is faithfully flat as a module over its endomorphism ring. Therefore, since $H_A(G)$ is torsion-free as an abelian group, the group $T_A H_A(G)$ is torsion-free, and G is A-solvable. By [4], an abelian group which is faithfully flat as an $E(A)$-module is projective with respect to sequences of A-solvable groups. Therefore, the induced sequence $H_A(A^n) \xrightarrow{H_A(\beta)} H_A(G) \to 0$ is exact. Since A has finite rank, every $E(A)$-module which is torsion-free as an abelian group is torsion-free as an $E(A)$-module [3]. In particular, $H_A(G)$ is free as a finitely generated torsion-free module over a PID; and the sequence $H_A(A^n) \xrightarrow{H_A(\beta)} H_A(G) \to 0$ splits. We obtain the commutative diagram

$$\begin{array}{ccccc}
T_A H_A(A^n) & \xrightarrow{T_A(\beta)} & T_A H_A(G) & \longrightarrow & 0 \\
{\scriptstyle l}\big\downarrow {\scriptstyle \theta_{A^n}} & & {\scriptstyle l}\big\downarrow {\scriptstyle \theta_G} & & \\
A^n & \xrightarrow{\beta} & G & \longrightarrow & 0
\end{array}$$

in which the top-row splits. This provides (b).

(b) $\Rightarrow$ (c). By (b), every finitely A-generated subgroup of a torsion-free reduced group G is A-projective. This shows that $S_A(G)$ is A-solvable proving (ii). In order to verify that A is a generalized rank 1 group, it is enough to show that $E(A)$ is semi-hereditary in view of [15]. For a finitely generated right ideal I of $E(A)$, we consider an exact sequence $E(A)^n \xrightarrow{\pi} I \to 0$ of right $E(A)$-modules. If $\delta\colon T_A(I) \to IA$ is the evaluation map, then $T_A(E(A)^n) \xrightarrow{\delta T_A(\pi)} IA \to 0$ is a split-exact sequence by (b). Therefore, the top-row of the commutative diagram

$$\begin{array}{ccccc}
H_A T_A(E(A)^n) & \xrightarrow{H_A(\delta) H_A T_A(\pi)} & H_A(IA) & \longrightarrow & 0 \\
\big\uparrow {\scriptstyle \phi_{E(A)^n}} & & \big\uparrow {\scriptstyle \phi} & & \\
E(A)^n & \xrightarrow{\pi} & I & \longrightarrow & 0
\end{array}$$

splits in which ϕ is the evaluation map.

(c) $\Rightarrow$ (a). We write $A = A_1 \oplus \cdots \oplus A_n$ where each A_i is indecomposable, and consider a pure non-zero subgroup U of A_i for some i. We can find pure subgroups V and W of A_i containing U such that V/U is divisible, W/U is reduced, and $A_i/U = V/U \oplus W/U$. Since $A_i/V \cong (A_i/U)/(V/U) \cong W/U$ is reduced, A_i/V is A-solvable by (c). Because the generalized rank 1 group A is flat as an $E(A)$-module, V is A-solvable, and $H_A(V)$ is a non-zero pure submodule of $H_A(A_i)$. But this is only possible if $H_A(V)$ is a direct summand of $H_A(A_i)$. Consequently, $V = A_i$, and A_i is a Murley group. In particular, every non-zero map $\alpha: A_i \to A_j$ is a monomorphism.

Suppose that i and j are indices with $\operatorname{Hom}(A_i, A_j) \neq 0$ but $\operatorname{Hom}(A_j, A_i) = 0$, and let $e_i: A \to A_i$ be the projection with kernel $\oplus_{k \neq i} A_k$. Then, $e_i E(A)\operatorname{Hom}(A_i, A_j)e_i = 0$, yields

$$0 \neq E(A)\operatorname{Hom}(A_i, A_j)e_i \subseteq N(E(A)),$$

which is not possible since generalized rank-1 groups have a semi-prime endomorphism ring. Therefore, A_i and A_j are quasi-isomorphic, hence isomorphic [13, p. 148] whenever $\operatorname{Hom}(A_i, A_j) \neq 0$. We can therefore rewrite the given decomposition of A as $A \cong B_1^{n_1} \oplus \cdots \oplus B_m^{n_m}$ where each B_i is an indecomposable Murley group, and $\operatorname{Hom}(B_i, B_j) = 0$ for $i \neq j$.

In order to show that each B_i is irreducible, we consider a prime p with $r_p(B_i) = 1$. The group $G = \oplus_{2^{\aleph_0}} B_i$ has p-rank $2^{\aleph_0}$ and contains a p-pure free subgroup X of rank $2^{\aleph_0}$ such that G/X is p-divisible. Since $\operatorname{Ext}(G/X, \hat{\mathbb{Z}}_p) = 0$, every map from X into $\hat{\mathbb{Z}}_p$ can be extended to G. This shows that $\hat{\mathbb{Z}}_p$ is B_i-generated. Moreover, every finitely A-generated subgroup of $\hat{\mathbb{Z}}_p$ is A-solvable by (c), and so $\hat{\mathbb{Z}}_p$ itself is A-solvable. Take an A-balanced epimorphism $F \to \hat{\mathbb{Z}}_p$ with kernel K where $F = \oplus_I B_i$ for some index-set I. If we apply $T_A H_A$ to the sequence $0 \to K \to F \to \hat{\mathbb{Z}}_p \to 0$, we obtain an A-balanced sequence $0 \to P \to F \to \hat{\mathbb{Z}}_p \to 0$ where P is A-projective. Since $S_{B_j}(P) = 0$ for all $j \neq i$, we have that P is B_i-projective. This shows that $\hat{\mathbb{Z}}_p$ is B_i-solvable. Since B_i is an indecomposable generalized rank 1 group, $E(B_j)b \cong E(B_i)$ for each non-zero $b \in B_i$.

If $r_0(E(B_i)) < r_0(B_i)$, then B_i has at least Goldie-dimension 2 as an $E(B_i)$-module. This is because $\mathbb{Q}B_i$ is a vector space over $\mathbb{Q}E(B_i)$ since $E(B_i)$ is semi-prime. Consequently we can find a monomorphism $\alpha: E(B_i) \oplus E(B_i) \to B_i$ of left $E(B_i)$-modules. Hence, $H_{B_i}(\hat{\mathbb{Z}}_p)^2$ is isomorphic to a subgroup of $T_{B_i}H_{B_i}(\mathbb{Z}_p)\hat{} \cong \hat{\mathbb{Z}}_p$. Since $H_{B_i}(\hat{\mathbb{Z}}_p) \cong \mathbb{Z}_p^l$ for some non-zero integer l, this is not possible. Therefore, B_i and $E(B_i)$ have the same rank and B_i is irreducible [19].

Finally, assume that $B_j/pB_j \neq 0$ for some j. Then, there exists a non-zero map $\phi: B_j \to \hat{\mathbb{Z}}_p$. Since $0 \to P \to F \to \hat{\mathbb{Z}}_p \to 0$ is A-balanced, ϕ induces a non-zero map from B_j into $F = \oplus_I B_i$, which is only possible if $i = j$.

Therefore, $E(A)/pE(A) \cong E(B_i^{n_i})/pE(B_i^{n_i}) \cong Mat_{n_i}(E(B_i)/pE(B_i)) \cong Mat_{n_i}(\mathbb{Z}/p\mathbb{Z})$ since B_i is a Murley group. This shows that A is p-simple. ∎

A group A is called (*finitely*) *faithful* if for any right ideal I of $E(A)$ (respectively, I is of finite index in $E(A)$), $IA \neq A$; and A is an $\mathcal{S}$*-group* if A generates each subgroup B of finite index in A. Arnold shows that A being a finitely faithful $\mathcal{S}$-group is equivalent to $\mathrm{Ext}(A, A)$ being torsion-free (Corollary 5.2 from [9]); and the strongly homogeneous finitely faithful $\mathcal{S}$-groups are precisely the groups of the form B^n for some irreducible Murley group B (Corollary 6.3 from [9]). Therefore, our last result gives an additional characterization of strongly homogeneous finitely faithful $\mathcal{S}$-groups:

COROLLARY 1.2. *The following conditions are equivalent for a torsion-free abelian group A of finite rank*:

(a) *A is a strongly homogeneous finitely faithful $\mathcal{S}$-group.*

(b) *$\mathbb{Q}E(A)$ is a simple algebra and every pure exact sequence $A^n \to G \to 0$ with G reduced splits.*

The conditions in part (a) of Theorem 1.1 simplify if A is completely decomposable:

COROLLARY 1.3. *The following conditions are equivalent for a completely decomposable group A of finite rank*:

(a) *Every pure exact sequence $A^n \to G \to 0$ in which G is reduced splits.*

(b) *If A_1 and A_2 are non-isomorphic rank 1 summands of A, then $A_1 + A_2 = \mathbb{Q}$.*

One of the important properties of a Butler group G is that $Bext^1(G, T) = 0$ for all torsion groups T. We shall see in Theorem 2.7 that $\mathcal{A}$-Butler groups, for certain classes $\mathcal{A}$, exhibit analogous behaviour. For the proof of Theorem 2.7 we need to know what the A-solvable torsion groups are in the case that A is an irreducible indecomposable Murley group. It is easy to see that irreducible groups are homogeneous, so the type of such a group is defined. We represent the type of an irreducible group A by a sequence (h_p) of non-negative integers and symbols ∞.

THEOREM 1.4. *Let A be p-simple for all primes p and of the form $A \cong B_1^{n_1} \oplus \cdots \oplus B_m^{n_m}$ where each B_i is an irreducible indecomposable Murley group of type $[(h_{i,p})]$ and rank at least 2. A torsion group G is A-solvable if and only if*

(i) *G is reduced, and*

(ii) *for every $i = 1, \ldots, m$, either $h_{i,p} = 0$ or $h_{i,p} = \infty$ for almost all primes p with $G_p \neq 0$.*

Proof. Suppose there is a non-zero A-solvable torsion group G which is not reduced. Let p be a prime such that $G[p] \neq 0$. Clearly $A/pA \neq 0$. There exists exactly one index $i \in \{1, \ldots, m\}$ with $B_i \neq pB_i$. Since B_i is an irreducible indecomposable Murley group with $r_0(B_i) > 1$, every pure subgroup V of B_i with rank 1 cokernel has the property that $B_i/V \cong \mathbb{Q}$. Hence, B_i contains a subgroup W such that $B_i/W \cong \mathbb{Z}(p^\infty)$. Then, as a direct summand of $G, \mathbb{Z}(p^\infty)$ is A-solvable.

By Theorem 2.1, A is a generalized rank 1 group. From [4], kernels of maps between A-solvable groups are A-solvable. Since B_i and B_i/W are A-solvable, the same holds for W. But $\operatorname{Hom}(B_j, B_i) = 0$ for $i \neq j$ yields $H_A(W) = H_{B_i}(W)$. Therefore, $H_{B_i}(W)$ is a non-zero ideal of $E(B_i)$ which is only possible if $E(B_i)/H_{B_i}(W)$ is bounded as an abelian group. Applying the functor T_A yields that $\mathbb{Z}(p^\infty) \cong T_A(E(B_i)/H_{B_i}(W))$ is bounded, a contradiction. Therefore, G is reduced.

For $i \in \{1, \ldots, m\}$, let F_i be a full free subgroup of B_i and write $G = \oplus_{j=1}^m G_j$ where $G_j = \oplus\{G_p \mid B_j \neq pB_j\}$. Since G is A-solvable, $H_A(G)$ is torsion as an abelian group, because otherwise $T_A(H_A(G)/tH_A(G)) = 0$ would yield a contradiction to the fact that generalized rank 1 groups are faithfully flat. Consequently, $\operatorname{Hom}(B_i/F_i, G_i)$ is torsion as a subgroup of $H_A(G)$. Since there always exists a non-zero homomorphism between two non-zero reduced p-groups, $(B_i/F_i)_p$ is divisible for all but finitely many primes p with $G_p \neq 0$ and $B_i \neq pB_i$. By a result of Warfield [6, Theorem 1.10], the structure of B_i/F_i determines the inner type $\operatorname{IT}(B_i) = [(t_{i,p})]$ of B_i in such a way that, for all but finitely many primes p with $G_p \neq 0$ and $B_i \neq pB_i$, we have $t_{i,p} = 0$. Since B_i is homogeneous, we obtain that $\operatorname{type}(B_i) = \operatorname{IT}(B_i)$, and so $h_{i,p} = 0$ for almost all primes p with $G_p \neq 0$ and $B_i \neq pB_i$.

Conversely, let G be a reduced torsion group as in the theorem, and write $G = \oplus_{j=1}^m G_j$ as in the previous paragraph. Because the class of A-solvable groups is closed under finite direct sums, it is enough to show that each G_i is A-solvable for each i. Since $\operatorname{Hom}(B_j, G_i) = 0$ for $i \neq j$, it is enough to show that G_i is B_i-solvable. Hence, we may assume that A is an irreducible indecomposable Murley group of type $[(h_p)]$. We select a pure rank 1 subgroup U of A, and consider the exact sequence $0 = \operatorname{Hom}(A/U, G) \to \operatorname{Hom}(A, G) \to \operatorname{Hom}(U, G)$ in which the first term vanishes since A/U is divisible. Since $h_p = 0$ for almost all primes p with $G_p \neq 0$, we have $\operatorname{Hom}(U, G)$ is torsion, and the same holds for $\operatorname{Hom}(A, G)$.

We want to remind the reader that a family $\{H_j \mid j \in J\}$ of abelian groups is A-small if, for every map $\alpha \in \operatorname{Hom}(A, \oplus_{j \in J} H_j)$, there is a finite subset J' of J such that $\alpha(A) \subseteq \oplus_{j \in J'} H_j$. While the class of A-solvable groups need not be closed with respect to arbitrary direct sums, it is closed with respect to direct sums of A-small families. Since we have just shown that $\{G_p \mid A \neq pA\}$ is an A-small family, it suffices to establish that all of the G_p are A-solvable. If p is a prime with $A \neq pA$, then $A/pA \cong \mathbb{Z}/p\mathbb{Z}$ provides that every bounded p-group is A-solvable by [2, Proposition 3.1], and hence, every finite subgroup of G is A-solvable. On the other hand, finitely A-generated p-groups are finitely cogenerated since A has finite rank. Therefore, the finitely A-generated subgroups of G_p are A-solvable, and the same holds for G_p. ∎

The requirement that $r_0(B_i) > 1$ in the last result is necessary since, if A has rank 1, then $\mathbb{Z}(p^\infty)$ is A-solvable for all primes p with $A \neq pA$. However, the arguments used in the last proof can easily be adopted to show

THEOREM 1.5. *Let A be a completely decomposable group which is p-simple for all primes p. Write $A \cong B_1^{n_1} \oplus \cdots \oplus B_m^{n_m}$ where each B_i is a rank 1 group of type $[(h_{i,p})]$. A torsion group G is A-solvable if and only if*

(i) *$G_p = 0$ for all primes p with $A = pA$, and*

(ii) *for every $i = 1, \dots, m$, either $h_{i,p} = 0$ or $h_{i,p} = \infty$ for almost all primes p with $G_p \neq 0$.*

COROLLARY 1.6. *Let A be p-simple for all primes p and of the form $A = A_1 \oplus A_2$ such that $A_1 \cong B_1^{n_1} \oplus \cdots \oplus B_m^{n_m}$ where each B_i is an indecomposable irreducible Murley group whose rank is at least 2 with $\operatorname{Hom}(B_i, B_j) = 0$ for $i \neq j$, and A_2 is completely decomposable. A torsion group G is A-solvable if and only if $G = G_1 \oplus G_2$ such that G_j is A_j-solvable for $j = 1, 2$.*

Proof. The corollary follows directly from the previous two theorems since A_1 and A_2 are fully invariant in A.

COROLLARY 1.7. *Let A be p-simple for all primes p and of the form $A \cong B_1^{n_1} \oplus \cdots \oplus B_m^{n_m}$ where each B_i is an indecomposable irreducible Murley group of idempotent type which has rank at least 2. A torsion group G is A-solvable if and only if it is reduced.*

2. A-BUTLER GROUPS

We extend the notion of a Butler group using the class of irreducible indecomposable Murley groups as a substitute for the rank one groups. A *balanced sequence of R-modules*, where R is an integral domain, is a sequence $0 \to L \to M \to N \to 0$ of R-modules relative to which each rank-1 R-module is projective. Similarly, given a class $\mathscr{A}$ of abelian groups, a sequence $0 \to G \to H \to K \to 0$ of abelian groups is *$\mathscr{A}$-balanced* if each member of $\mathscr{A}$ is projective relative to the sequence. We want to remind the reader that a group is finitely $\mathscr{A}$-decomposable if it is of the form $A_1 \oplus \cdots \oplus A_n$ for some $A_1, \dots, A_n \in \mathscr{A}$.

DEFINITION 2.1.

(a) Let R be an integral domain. A Butler R-module is any torsion-free R-module which is the balanced image of a finite rank completely decomposable R-module.

(b) Given a class $\mathscr{A}$ of abelian groups, an $\mathscr{A}$-Butler group is any $\mathscr{A}$-balanced image of a finitely $\mathscr{A}$-decomposable group.

We assert that the standard characterizations of Butler groups carry over to Butler R-modules if R is a PID. In the following, the symbol Q represents the quotient field of the integral domain R, and $\overline{R}$ denotes the integral closure of R in Q.

PROPOSITION 2.2. *The following conditions are equivalent for a subring R of an algebraic number field.*

(a) *Every rank-1 R-module is of the form $I \otimes X$ for some ideal I and a rank-1 torsion-free abelian group X.*

(b) *pR is a primary ideal for every integral prime p such that $pR \neq R$.*

Proof. (a) $\Rightarrow$ (b). Since $\mathbb{Q}R$ is a finite dimensional integral domain, it is a field, from which it follows that every ideal of R has finite index. Therefore every non-zero prime ideal of R is maximal, and R is noetherian. Consequently, a non-zero ideal I of R is primary precisely when I is contained in exactly one maximal ideal of R. Suppose that there are distinct maximal ideals P_1 and P_2 over pR. Then, R_{P_1} and R_{P_2} are p-local rank-1 R-modules since $q \notin P_i$ for any integral prime $q \neq p$ yields that q is invertible in R_{P_i}. By (a), we can find ideals I_1 and I_2 of R such that $R_{P_i} = I_i \otimes \mathbb{Z}_p$ for $i = 1, 2$. Since I_1 and I_2 are quasi-equal, the same holds for R_{P_1} and R_{P_2}. In particular, $R_{P_1} R_{P_2} \doteq R_{P_1}$ as submodules of $Q = \mathbb{Q}R$. But this contradicts $R_{P_1} R_{P_2} = Q$ [16, p. 29].

(b) $\Rightarrow$ (a). Let M be a rank-1 R-module such that $R \subseteq M \subseteq Q = \mathbb{Q}R$, and consider rank-1 pure subgroups X and Y of M. Then, $M/RX \subseteq Q/RX$ is torsion, and we can find $r \in R$ such that $rX \cap Y \neq 0$. In as much as Y is pure in M and X has rank 1, we obtain $rX \subseteq Y$. Since $R \subset E_{\mathbb{Z}}(M)$, this yields $\operatorname{type}(X) \leq \operatorname{type}(Y)$, and M is homogeneous as an abelian group. In particular, M is X-solvable [20, Proposition 1] and $M \cong H \otimes X$. We show that $H = H_X(M)$ is isomorphic to an ideal of $R_\pi = \bigcap_{p \in \pi} R_p$ where $\pi = \{p \mid pM \neq M\}$. This will complete the proof of the proposition since the ideals of R_π are of the form $I \otimes \mathbb{Z}_\pi$ for some ideal I of R. Without loss of generality, we may assume $R = R_\pi$.

Set $\overline{M} = \overline{R}M$ which is a rank-1 $\overline{R}$-module containing M as a submodule of finite index since $n\overline{R} \subseteq R$ for some non-zero integer n. Then, $\overline{H} = H_X(\overline{M})$ contains H as an R-submodule of finite index. Suppose we have shown that $m\overline{H} \subseteq \overline{R}$ for some non-zero integer m. (Observe that the assumption $R_\pi = R$ yields $\overline{R}_\pi = \overline{R}$.) Then, $nmH \subseteq nm\overline{H} \subseteq n\overline{R} \subseteq R$, and $H \cong nmH$, an ideal of R.

Since M is a homogeneous group whose type is type(X) and $R = R_\pi$, the group H is homogeneous and its type is type$(E(X))$ = type(R). (It is easy to see that R is homogeneous.) Hence, $\overline{H}$ also is homogeneous as an abelian group and its type is type$(\overline{R})$ = type(R). Since $\overline{R}$ is Dedekind domain, one can also consider the types of rank one modules over $\overline{R}$. This module type determines the module up to quasi-isomorphism of $\overline{R}$-modules.

To compute the type of $\overline{H}$ as an $\overline{R}$-module, we select a non-zero $h \in \overline{H}$. As with $\mathbb{Z}$, the type of $\overline{H}$ as an $\overline{R}$-module is the equivalence class of the height characteristic $\chi^{\overline{H}}(h)$: spec$(\overline{R}) \to \mathbb{N} \cup \{\infty\}$ where $[\chi^{\overline{H}}(h)](P) = n$ provided $h \in P^n\overline{H} \setminus P^{n+1}\overline{H}$ and $[\chi^{\overline{H}}(h)](P) = \infty$ when no such n exists. Suppose that $\chi^{\overline{H}}(h)$ is not equivalent to $\chi^{\overline{R}}(1)$. Then $[\chi^{\overline{H}}(h)](P) > 0$ for infinitely many $P \in$ Spec$(\overline{R})$. Since $\overline{R}/R$ is finite, there are only finitely many integral primes p for which $\overline{R}_p \neq R_p$; let $\mathscr{I}$ be the set of these primes. When $p \notin \mathscr{I}$ and $p\overline{R} \neq \overline{R}$, $pR \neq R$ and so pR is primary by our assumption. But $p\overline{R}_p = pR_p$ has only one maximal ideal lying over it, from which we conclude that $p\overline{R}$ has a unique maximal ideal over it (i.e., $p\overline{R}$ is primary).

It is a standard result from number theory that almost all integral primes p with $p\overline{R} \neq \overline{R}$ are unramified; meaning that for almost all integral primes p, P^2 does not divide $p\overline{R}$ for all $P \in$ Spec$(\overline{R})$. Combining these observations, we see that almost all primes in $\{p \mid p\overline{R} \neq \overline{R}\}$ are inert, i.e., remain prime in $\overline{R}$. Hence, there are only finitely many primes $p_1, \dots, p_m$ with $p_j\overline{R} \neq \overline{R}$ for which either $p_j \in \mathscr{I}$, or, $p_j\overline{R}$ is primary but not prime (of course, there may be no primes with this property).

Let $\mathscr{S} = \{P \in$ Spec$(\overline{R}) \mid P$ lies over some p_j, $j = 1, \dots, m\}$ and note that this set is finite. Thus,

$$\mathrm{spec}(\overline{R}) = \left\{ p\overline{R} \mid p\overline{R} \neq \overline{R} \text{ and } p \neq p_1, \dots, p_m \right\} \cup \mathscr{S},$$

because every prime ideal of $\overline{R}$ lies over some integral prime. Since $[\chi^{\overline{H}}(h)](P) > 0$ for infinitely many primes in Spec$(\overline{R})$, we obtain $h \in p\overline{H}$ for infinitely many integral primes p. But this contradicts the fact that $\overline{H}$ is a homogeneous group with idempotent type. Thus, $\overline{H}$ has the same type as $\overline{R}$ as an $\overline{R}$-module which implies that $\overline{H}$ is a fractional ideal of $\overline{R}$. ∎

If the additive group of R is an indecomposable Murley group, then R is a PID such that pR is maximal (hence primary) in R whenever $pR \neq R$.

COROLLARY 2.3. *If R is a ring such that R^+ is an indecomposable Murley group, then the rank-*1 *R-modules are of the form $R \otimes X$ for some subgroup X of $\mathbb{Q}$.*

Given a torsion-free ring R of finite rank and R-modules M and N, it was shown in [14] that $\mathrm{Hom}(M, N) = \mathrm{Hom}_R(M, N)$ provided there is a prime p such that $p^{\omega}R = 0$, $r_p(R) = 1$, $r_p(M) < \infty$, and $p^{\omega}N = 0$. Using this result, we can establish

LEMMA 2.4. *Let R be a ring such that R^+ is an indecomposable Murley group. If M and N are finite rank torsion-free R-modules such that N is reduced, then* $\mathrm{Hom}(M, N) = \mathrm{Hom}_R(M, N)$.

Proof. We have $p^{\omega}R = 0$ and $r_p(R) = 1$ for every prime p with $pR \neq R$. By the remark preceding the lemma, we have $\mathrm{Hom}(M, N/p^{\omega}N) = \mathrm{Hom}_R(M, N/p^{\omega}N)$ for all these primes. Setting $\pi = \{p \mid pR \neq R\}$, since N is reduced, we obtain an exact sequence $0 \to N \to \prod_{p \in \pi} N/p^{\omega}N$; which in turn induces the sequence

$$0 \to \mathrm{Hom}(M, N) \to \mathrm{Hom}\big(M, \textstyle\prod_{p \in \pi} N/p^{\omega}N\big).$$

But,

$$\begin{aligned}\mathrm{Hom}\Big(M, \prod_{p \in \pi} N/p^{\omega}N\Big) &= \prod_{p \in \pi} \mathrm{Hom}(M, N/p^{\omega}N)\\ &= \prod_{p \in \pi} \mathrm{Hom}_R(M, N/p^{\omega}N)\end{aligned}$$

which shows that every map in $\mathrm{Hom}(M, \prod_{p \in \pi} N/p^{\omega}N)$ is R-linear. In particular, $\mathrm{Hom}(M, N) = \mathrm{Hom}_R(M, N)$.

LEMMA 2.5. *Let R be a ring such that R^+ is an indecomposable Murley group. If M is a reduced torsion-free group of finite rank which is R-generated, then M is an R-module.*

Proof. Let $\phi\colon F \to M$ be an epimorphism where $F = \oplus_{\delta} R$. For $x \in M$ and $r \in R$, select $y \in F$ with $\phi(y) = x$ and set $rx = \phi(ry)$. If $x = \phi(y_1) = \phi(y_2)$, define $f\colon R \to M$ by $f(s) = \phi(sy_1) - \phi(sy_2)$ where $s \in R$. Then, $1 \in \ker f$, and $R/\langle 1 \rangle_*$ is divisible. Thus, $f = 0$ since M is reduced, and $\phi(ry_1) = \phi(ry_2)$. A standard argument shows that the above defines an R-module structure on M. ∎

Evidently, $R/\langle 1 \rangle_*$ is R-generated but not an R-module provided $R \not\subseteq \mathbb{Q}$; so the assumption that M be reduced in Lemma 2.5 is critical.

When R is a ring whose additive group is an indecomposable Murley group, we use the symbol $\mathscr{A}_R$ to denote the family of rank-1 R-modules. By

Corollary 2.3, the elements of $\mathscr{A}_R$ are of the form $R \otimes X$ for some subgroup X of $\mathbb{Q}$. If X has type τ, then we write A_τ for $R \otimes X$. We require one last lemma before presenting our main result of this section.

LEMMA 2.6. *If A is an indecomposable Murley group, then any finitely A-generated A-projective is A-free.*

Proof. If $P \oplus P' \cong A^n$ for some n, then $H_A(P) \cong E(A)^m$ for some m since $E(A)$ is a PID. Thus, $P \cong A^m \cong T_A H_A(P)$. ∎

The main result of this paper can now be substantiated.

THEOREM 2.7. *Let R be a ring such that R^+ is an indecomposable Murley group. The following conditions are equivalent for a reduced torsion-free group G of finite rank*:

(a) *G is an $\mathscr{A}_R$-Butler group.*

(b) *G is an R-module that is also a Butler R-module.*

(c) *G is the image of $A_1 \oplus \cdots \oplus A_n$ for some $A_1, \dots, A_n \in \mathscr{A}_R$.*

(d) *G is a pure $\mathscr{A}_R$-generated subgroup of $A_1 \oplus \cdots \oplus A_n$ for some $A_1, \dots, A_n \in \mathscr{A}_R$.*

(e) *G is an R-generated group such that*

(i) *Typeset(G) is finite.*

(ii) *For every type τ, $G(\tau) = (G^*(\tau))_* \oplus P_\tau$ for some A_τ-free group P_τ.*

(iii) *$(G^*(\tau))_* / G^*(\tau)$ is finite for all types τ.*

(f) *$S_R(G) = G$ and an $\mathscr{A}_R$-balanced exact sequence $0 \to T \to B \to G \to 0$ splits whenever T is a reduced torsion-group with $T_p = 0$ if $R = pR$.*

Proof. (a) ⇒ (b). There exist $A_1, \dots, A_n \in \mathscr{A}_R$ and an $\mathscr{A}_R$-balanced sequence $0 \to K \to \bigoplus_{j=1}^n A_j \xrightarrow{\phi} G \to 0$. By Lemmas 2.4 and 2.5, G has an R-module structure which makes ϕ into an R-module map. Let A be a rank-1 R-module. Condition (a) yields that the induced sequence $0 \to \operatorname{Hom}(A, K) \to \operatorname{Hom}(A, \oplus_j A_j) \to \operatorname{Hom}(A, G) \to 0$ is exact. By Lemma 2.4, the homomorphism groups in this sequence coincide with their corresponding subgroups of R-homomorphisms proving (b). A similar argument can be used to establish the converse of the previous implication. Observe that the proofs for (b) ⇒ (c), (d) are obvious. To verify (c) ⇒ (b), we consider a group homomorphism $A_1 \oplus \cdots \oplus A_n \xrightarrow{\phi} G \to 0$. As before, G carries an R-module structure which makes ϕ an R-module map. Since R is a PID, G is a Butler R-module.

(d) ⇒ (b). Since G is R-generated, it is an R-module by Lemma 2.5, and any pure embedding of G into $N = A_1 \oplus \cdots \oplus A_n$ as abelian groups is an

R-module map (Lemma 2.4). We now show that G is a pure R-submodule of N. As noted earlier, as R is irreducible and Murley, R is strongly homogeneous. From Arnold's characterization of strongly homogeneous rings [8], any element of R is an integral multiple of a unit of R. If $x \in G \cap rN$ for some $r \in R$, then because R is strongly homogeneous, $r = ku$ for some integer k and some unit u. From the fact that G is a pure subgroup of N, we have $k^{-1}x \in G \cap uN$, and so $u^{-1}k^{-1}x \in G$. Therefore, $x \in nuG = rG$. So G is a pure R-submodule of a finite rank completely decomposable R-module, and is therefore a Butler R-module since R is a PID.

(a) ⇒ (e). Since G is an $\mathscr{A}_R$-Butler module, we obtain an $\mathscr{A}_R$-balanced exact sequence $0 \to K \to C \xrightarrow{\beta} G \to 0$ where C is a completely decomposable R-module of finite rank. By Lemma 2.4, β is an R-module map. Because finitely generated torsion R-modules are finite (R has p-rank at most one for all primes p), (e) results using the arguments from [11] because R is a PID. The implication (e) ⇒ (a) follows from [11] for this reason as well.

(a) ⇒ (f). By Lemma 2.4, we have $H_R(M) = \mathrm{Hom}_R(R, M) = M$ if M is a torsion-free reduced R-module of finite rank. Therefore, we may identify M, $H_R(M)$, and $T_R(M)$, and replace θ_M and $\phi_{H_R(M)}$ by the appropriate identity maps.

Consider an $\mathscr{A}_R$-balanced sequence $0 \to T \xrightarrow{\alpha} B \xrightarrow{\beta} G \to 0$. It induces the sequence $0 \to H_R(T) \xrightarrow{H_R(\alpha)} H_R(B) \xrightarrow{\beta_*} G \to 0$ in which $H_R(T)$ is torsion. To see this, observe that T is A-solvable by Theorem 1.4 since R has idempotent type. Let A be a rank-1 R-module, and consider an R-module map $\phi\colon A \to G$, to obtain the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & T_R H_R(T) \arrow[r, "T_R H_R(\alpha)"] \arrow[d, "l"', "\theta_T"] & T_R H_R(B) \arrow[r, "\beta_{**}"] \arrow[d, "\theta_B"] & G \arrow[r] \arrow[d, "1_G"] & 0 \\
0 \arrow[r] & T \arrow[r, "\alpha"] & B \arrow[r, "\beta"] & G \arrow[r] & 0
\end{tikzcd}$$

in which θ_T is an isomorphism by Theorem 1.4. We can find a map $\lambda\colon A \to B$ with $\beta\lambda = \phi$. Using the above mentioned identifications, we obtain $\beta_* H_R(\lambda) = \phi$. Since G is a Butler R-module by the equivalence of the first four conditions in the theorem, the sequence $0 \to H_R(T) \xrightarrow{H_R(\alpha)} H_R(B) \xrightarrow{\beta_*} G \to 0$ splits, and so does the top-row of the diagram. But then, the original sequence splits.

(f) ⇒ (c). In order to show that G is an $\mathscr{A}_R$-Butler group, we show that G is a Butler R-module. For this, we consider an R-balanced exact sequence $0 \to T \to M \to G \to 0$ where T is a torsion module. Since G is reduced, G does not contain a torsion-free divisible summand. Consequently, we may assume that T is reduced. A slight modification of the arguments used in the previous part of the proof yields the splitting of the given sequence. ∎

COROLLARY 2.8. *Let R be a ring whose additive group is an indecomposable Murley group. An $\mathcal{A}_R$-Butler group with linearly ordered typeset is $\mathcal{A}_R$-decomposable.*

Proof. If G is an $\mathcal{A}_R$-Butler group with linearly ordered typeset, then G is a Butler R-module with linearly ordered R-module typeset, and since R is PID, G is isomorphic to a direct sum of rank-1 R-modules. ∎

Our next result shows that our attempt to preserve a few basic properties of classical Butler groups forces us to restrict our discussion to $\mathcal{A}_R$.

PROPOSITION 2.9. *Let $\mathcal{A}$ be a pseudo-rigid family of indecomposable reduced torsion-free abelian groups of finite rank which contains a minimal element A_0 in the sense that each $A \in \mathcal{A}$ is A_0-generated. The following conditions are equivalent*:

(a) *A_0 is an irreducible Murley group, and $\mathcal{A} \subseteq \{X \otimes A_0 \mid X \subseteq \mathbb{Q}\}$.*

(b) *Whenever $\mathcal{A}'$ is a finite subset of $\mathcal{A}$ and $0 \to U \to C \to G \to 0$ is an exact sequence of torsion-free groups of finite rank with C $\mathcal{A}'$-decomposable, then*

(i) *U is an $\mathcal{A}'$-Butler group if and only if it is A_0-generated,*

(ii) *$G = \mathbb{Q}^m \oplus G'$ for some $m < \omega$ and some $\mathcal{A}'$-Butler group G',*

and

(iii) *If $|\mathcal{A}'| = 1$ and G is reduced, then U is a direct summand of C.*

Proof. Observe that (a) ⇒ (b) is a direct consequence of Theorem 2.7.

(b) ⇒ (a). Since the groups in $\mathcal{A}$ are indecomposable, Theorem 1.1 yields that the elements of $\mathcal{A}$ are irreducible Murley groups. Now consider $B \in \mathcal{A}$ and a non-zero map $\alpha: A_0 \to B$. By (iii), $\ker \alpha$ is a direct summand of A_0 since B is reduced. Therefore, α is a monomorphism, and $r_0(A_0) \leq r_0(B)$. Since B is an A_0-solvable group by Theorem 1.1, [3] yields that the $\mathbb{Z}$-purification $\alpha(A_0)_*$ of $\alpha(A_0)$ in B is again A_0-generated. However, condition (i) in (b) shows that $\alpha(A_0)_*$ is a B-Butler group. Therefore, $\alpha(A_0)_*$ is B-generated, and we can find a non-zero map $\beta: B \to \alpha(A_0)_*$ which is a monomorphism since it is a non-zero endomorphism of the indecomposable irreducible Murley group B. Consequently, $r_0(B) \leq r_0(\alpha(A_0)_*) = r_0(A_0)$, and there is an exact sequence $0 \to A_0 \xrightarrow{\alpha} B \to T \to 0$ with T torsion. It induces the exact sequence $0 \to H_{A_0}(A_0) \xrightarrow{\alpha} H_{A_0}(B) \to M \to 0$ for some $E(A_0)$-submodule M of $H_{A_0}(T)$. Since B is A_0-solvable, the additive group of M has to be torsion since otherwise $B/\alpha(A_0) \cong T_{A_0}(M)$ would not be torsion because A_0 is a faithful $E(A_0)$-module. This shows that $H_{A_0}(B)$ is a rank-1 module over $E(A_0)$, and there is a subgroup X of $\mathbb{Q}$ with $H_{A_0}(B) = X \otimes E(A_0)$ by Corollary 2.3. But then, $B \cong T_{A_0} H_{A_0}(B) \cong X \otimes A_0$ as desired. ∎

As a main application of Theorem 2.7 we obtain the following structure theorem for Butler R-modules.

THEOREM 2.10. *Let R be indecomposable and Murley. The following conditions are equivalent for a torsion-free reduced group M of finite rank*:

(a) *M is a Butler R-module.*

(b) *$M = R \otimes B$ for some Butler group B.*

Proof. (b) ⇒ (a). Let C be a completely decomposable group of finite rank mapping onto B. Then, $R \otimes C$ is an $\mathscr{A}_R$-decomposable group which maps onto $M = R \otimes B$. Thus, M is an $\mathscr{A}_R$-Butler group by Theorem 2.7.

(a) ⇒ (b). Let N be a completely decomposable R-module of finite rank which fits into an exact sequence $N \xrightarrow{\phi} M \to 0$ of R-modules. By Proposition 2.2, $N = \bigoplus_{j=1}^n R \otimes X_j$ for subgroups $X_1, \dots, X_n$ of $\mathbb{Q}$. Since $S = R \cap \mathbb{Q}$ is a pure subring of R, we obtain that $C = \bigoplus_{j=1}^n S \otimes X_j$ is a pure subgroup of N with $R \otimes C \cong RC = N$. The subgroup $B = \phi(C)$ of M is a classical Butler group.

We consider the commutative square

$$\begin{array}{ccccc}
R \otimes C & \longrightarrow & R \otimes B & \longrightarrow & 0 \\
{\scriptstyle nat}\big\downarrow{\scriptstyle l} & & \big\downarrow{\scriptstyle \mu} & & \\
N & \xrightarrow{\phi} & M & \longrightarrow & 0
\end{array}$$

where $\mu(r \otimes b) = rb$ is an epimorphism. We aim to show that μ is an isomorphism and do so by considering p-adic localizations in order to deduce that μ is one-to-one. Let K denote the kernel of μ. With $\hat{\mathbb{Z}}_p$ equal to the ring of p-adic integers (as before), we use $\hat{L}_p$ generally to denote $\hat{\mathbb{Z}}_p \otimes L$.

Tensor the sequence $0 \to K \to R \otimes B \to M \to 0$ with $\hat{\mathbb{Z}}_p$ to obtain $0 \to \hat{K}_p \to \widehat{(R \otimes B)}_p \to \hat{M}_p \to 0$. By the local/global theory [13], $K = \bigcap_p \hat{K}_p$. Now B is contained as a pure subgroup in $R \otimes B$ (under the identification with $S \otimes B$) and μ is the identity map from B into M. Passing to localizations, $\hat{\mu}_p|_{\hat{B}_p}: \hat{B}_p \to \hat{M}_p$ is the identity map as well. Since R has p-rank one, and $p^\omega R = 0$, R embeds as a subring of $\hat{\mathbb{Z}}_p$. Passing to completions, $\hat{R}_p = \hat{\mathbb{Z}}_p$. Therefore $\widehat{(R \otimes B)}_p = \hat{B}_p$ and $\hat{\mu}_p$ is an isomorphism. This implies that $K = \ker \mu$ is zero as desired. ∎

REFERENCES

1. U. Albrecht, Baer's Lemma and Fuchs' Problem 84a, *Trans. Amer. Math. Soc.* **293** (1986), 565–582.
2. U. Albrecht, Abelian groups A such that the category of A-solvable groups is preabelian, *Contemp. Math.* **87** (1989), 117–131.
3. U. Albrecht, Endomorphism rings and a generalization of torsion-freeness and purity, *Comm. Algebra* **17**, No. 5 (1989), 1101–1135.
4. U. Albrecht, Endomorphism rings of faithfully flat abelian groups, *Results Math.* **17** (1990), 179–201.
5. U. Albrecht, On direct summands of $\mathscr{A}$-separable R-modules, *Forum Math.* **2** (1990), 103–117.
6. D. M. Arnold, "Finite Rank Torsion-Free Abelian Groups and Rings," Lecture Notes in Math., Vol. 931, Springer-Verlag, New York/Berlin, 1982.
7. D. M. Arnold, Endomorphism rings and subgroups of finite rank torsion-free Abelian groups, *Rocky Mountain J. Math.* **12** (1982), 241–256.
8. D. M. Arnold, Strongly homogeneous torsion-free groups, *Proc. Amer. Math. Soc.* **56** (1976), 67—72.
9. D. M. Arnold, Pure subgroups of finite rank completely decomposable groups, *in* "Abelian Group Theory," Lecture Notes in Math., Vol. 874, Springer-Verlag, New York/Berlin, 1981.
10. D. Arnold, R. Hunter, and F. Richman, Global Azumaya theorems in additive categories, *J. Pure Appl. Algebra* **16** (1980), 232–242.
11. D. M. Arnold and C. Vinsonhaler, Pure subgroups of finite rank completely decomposable groups, II, *in* Lecture Notes in Math., Vol. 1006, pp. 97–143, Springer-Verlag, New York/Berlin, 1983.
12. D. M. Arnold and C. Vinsonhaler, Finite rank Butler groups: A survey of recent results, *in* "Abelian groups," Lecture Notes in Pure and Applied Mathematics, Vol. 146, Dekker, New York, 1991.
13. L. Fuchs, "Infinite Abelian Groups," Vol. II, Academic Press, San Diego, 1972.
14. H. Pat Goeters, An extension of Warfield duality for abelian groups, *J. Algebra* **180** (1996), 848–861.
15. M. Huber and R. Warfield, Homomorphisms between Cartesian powers of an Abelian group, *in* "Oberwolfach Proceedings, 1981," Lecture Notes in Math., Vol. 874, pp. 202–227, Springer-Verlag, New York/Berlin, 1981.
16. E. Matlis, "Torsion-Free Modules," Univ. of Chicago Press, Chicago, 1972.
17. C. E. Murley, The classification of certain classes of torsion-free abelian groups of finite rank, *Pacific J. Math.* **40** (1972), 647–665.
18. K. Rangaswamy, On C-separable abelian groups, *Comm. Algebra* **13** (1985), 1219–1227.
19. J. D. Reid, "On the Ring of Quasi-Endomorphisms of a Torsion-Free Group," Scott, Foresman, Chicago, 1963.
20. R. B. Warfield, Homomorphisms and duality for torsion-free abelian groups, *Math. Z.* (1968).